Trading mechanism selection with budget constraints

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Abstract: We present an equilibrium search model of competing mechanisms where some buyers are budget constrained. Absent budget constraints, the existing literature capitulates that if buyers differ in their valuations then in the unique equilibrium all sellers hold second price auctions (e.g. McAfee (1993)) whereas if buyers are homogeneous then sellers are indifferent across a large number of payoff-equivalent mechanisms (e.g. Eeckhout and Kircher (2010)). We show that these results are not robust to the presence of budget constrained buyers; merely lowering the budgets of a few buyers renders the auction equilibrium as well as payoff equivalence unsustainable. If buyers differ only slightly in terms of their ability to pay then sellers prefer fixed price trading; otherwise they prefer auctions.

Keywords: Trading Mechanisms, Budget Constraints, Competitive Search

JEL: C78, D4, D83

1 Introduction

We study the trading mechanism selection and the performance of various trading mechanisms in a model of competitive search, where some buyers are budget constrained. The adoption of a particular mechanism is a strategic decision in that it signals how the seller intends to share the surplus ex-post, which in turn influences the attractiveness of the store and pins down the expected demand. Mechanism selection becomes more strategic if potential customers have limited budgets. Indeed sellers often face buyers who are willing to pay but have limited immediate financial resources to do so. Casual observations suggest that markets for houses, automobiles and other expensive durable goods (appliances, electronic equipments, furniture, business equipments, etc.) often exhibit this trait. Despite its practical importance little attention has been paid to the relationship between buyers' limited purchasing power and the sale mechanism in place.[2]

Absent budget constraints, trading mechanism selection has been studied extensively. In a competitive search setting where buyers differ in their valuations, McAfee [8] shows that the unique

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[2]A number of articles study the presence of budget constrained buyers from a mechanism design perspective. For instance, Che and Gale [4] obtains an optimal mechanism to sell to a budget constraint buyer; Zheng [15] studies auctions where budget-constrained bidders may declare bankruptcy. We differ from this literature by introducing competing sellers.

equilibrium entails all sellers holding second price auctions with a reserve and buyers randomizing across stores. Peters [9] generalizes McAfee's result with heterogeneous sellers. When buyers are identical in their valuations, payoff equivalence emerges. Kultti [7] demonstrates that price posting is payoff equivalent to auctions; Eeckhout and Kircher [5] show that payoff equivalence is not specific to these two mechanism, rather it holds for all 'payoff complete' mechanisms that included fixed price trading and auctions.[3]

We demonstrate that these results are not robust to the presence of budget constrained buyers. For instance, the auction equilibrium in McAfee [8] fails to exist if low budget customers differ only slightly from high budget customers: a seller can unilaterally do better than holding an auction if he chooses fixed price trading. The reason is this. In a bidding contest high types always win; a low type can only purchase if there is no high type present in the auction and even then he faces the prospect of competing against other low types. So, if a seller switches to fixed price trading, which is egalitarian at the point of service, then he can attract low types almost for sure and improve his payoff. Of course there needs to be 'enough room' so that the seller can post a profit maximizing price; that is the budgets of low types need to be sufficiently high. Such deviation is possible even when there are very few low types present in the market. So, merely lowering the budgets of a few buyers renders the auction equilibrium in [8] unsustainable against price posting. Payoff equivalence results disappear because of similar reasons.

We characterize outcomes where sellers trade via price posting and via auctions and then we provide sufficient conditions under which either mechanism emerges in equilibrium. With fixed-price trading there are three possibilities:

1. Customers differ only slightly in terms of their ability to pay. The budget constraint is slack; so all sellers post affordable prices and no screening takes place.

2. Low types have severely limited budgets, so, no seller trades with them. Only high types can afford to shop; low types are completely screened out ex-ante.

3. Customers differ moderately. In this case some stores choose to be expensive and trade with high types while other stores remain affordable and trade with low types. Cheap stores are more crowded and possess a higher trade risk—the risk of not being able to purchase. This is why high budget customers strictly prefer expensive stores and avoid shopping at cheap stores.

With second price auctions sellers do not attempt to screen out low types ex-ante (posting an unaffordable reserve price effectively prevents low types from participating in the auction); screening occurs ex-post. Being expensive and catering to high types pays off if the expensive store possesses some distinct advantage. With fixed price trading there is such an advantage: a higher probability of service. But when the trading mechanism is an auction then high types are not worried about by the presence of low types since they can outbid them. The advantage disappears hence ex-ante screening does not take place.

The decision between auctions and fixed price trading boils down to the degree of heterogeneity across customers. If they differ only slightly in terms of their ability to pay then sellers prefer

[3]The literature on competing mechanisms may be divided into two categories. The first category focuses on a monopolist seller who selects a trading mechanism in order to maximize his expected profit. The set of alternatives typically includes auctions, bargaining and price posting. The approach is 'partial equilibrium' in that the demand is taken as given and mechanisms feature exogenous costs. Among others see [13, 14] and the references therein. The second category, to which this paper belongs, has competitive environments where demand at a store endogenously depends on the trading mechanism in place; for instance see [5, 6, 7, 8, 9, 12]. The main difference between our paper and the preceding articles is the presence of budget constrained buyers.

fixed price trading, otherwise they prefer auctions. The intuition is same as above. In terms of efficiency both outcomes are constrained efficient. With risk neutral buyers and sellers efficiency is synonymous with maximizing the number of trades. This can be achieved if ex-ante screening does not take place, i.e. if all sellers cater to both types of customers. With fixed price trading ex-ante screening may occur, however the outcome that survives the availability of auctions is the one without screening. With auctions ex-ante screening never occurs; so, in either case the resulting outcome is efficient.

2 Model

Consider an economy with a large number of risk-neutral buyers and sellers, where the buyer-seller ratio is $\lambda > 0$. Each seller is endowed with one unit of a good and wants to sell at a price above his reservation price of zero. Similarly each buyer wants to purchase one unit of an indivisible good and is willing to pay up to his reservation price, which is normalized to one. Buyers are identical in terms of their valuation of the good but they differ in terms of their ability to pay. A fraction l of buyers (low types) can pay up to $b < 1$ whereas the rest (high types) can pay pay up to 1. The parameters l and b are common knowledge; let $\overline{\lambda} = \lambda\,(1 - l)$ and $\underline{\lambda}=\lambda l$. The current setup has buyers who are homoegenous in their valuations; in Appendix II we extend the model by considering buyers with different valuations.

The game proceeds in two stages. In the first stage sellers simultaneously and independently choose a mechanism $m \in \mathbb{M}$ and a reserve price $r \in [0, 1]$. The pair (m, r) pins down the sale price $p_{m,\overline{n},\underline{n}}(r)$ for every possible demand realization, where $\overline{n}$ and $\underline{n}$ denote the number of high and low type customers at a store. In the second stage buyers observe sellers' selections and choose one store to visit; however once they reach a store they cannot move elsewhere. If the customer is alone at the store then he pays the reserve price and obtains the good for sure; however in case of excess demand the sale price and the probability of service depend on $\overline{n}$ and $\underline{n}$ as well as sale mechanisms in place. If trade takes place at price p then the seller realizes payoff p and the buyer $1 - p$. Those who do not trade earn zero. Once players realize their gains the game ends.

Demand Distribution. We focus on strongly symmetric outcomes, where, on and off the equilibrium path, buyers of the same type direct their search randomly and identically across stores. In a large economy with infinitely many buyers and sellers the distribution of demand across stores is i.i.d. with Poisson arrival rates $\overline{\lambda}_m$ and $\underline{\lambda}_m$. We refer to these parameters as the queue lengths consisting of high and low types. So, the probability that a seller meets exactly $\overline{n}$ high type buyers equals to

$$z_{\overline{n}}(\overline{\lambda}_m) = \frac{e^{-\overline{\lambda}_m}\overline{\lambda}_m^{\overline{n}}}{\overline{n}!} \text{ for } \overline{n} = 0, 1, ... \tag{1}$$

The probability of meeting low types is likewise. The queue lengths are endogenous and are determined via indifference conditions (3). Letting γ_m denote the fraction of stores trading via mechanism m we have

$$\sum_{m\in\mathbb{M}} \gamma_m\underline{\lambda}_m = \underline{\lambda} \quad\text{and}\quad \sum_{m\in\mathbb{M}} \gamma_m\overline{\lambda}_m = \overline{\lambda}. \tag{2}$$

Screening. The reserve price may be used as an ex-ante screening device. A seller who wishes to trade with high types only can do so by posting $r > b$. We assume that sellers can use cash bonds or financial disclosure requirements to implement ex-ante screening.[4]

[4]Buyers pay upfront a sum equal to the reserve price to a third party. In case the buyer obtains the good, the

The type of a buyer does not affect his likelihood of meeting a seller (assuming, of course, the reserve price is affordable). The probability of service, however, might be affected by the type. Let $\underline{q}_m(\overline{n}, \underline{n})$ denote the probability of service for a low type customer and let $\overline{q}_m$ be likewise. With auctions we have

$$\underline{q}_a(\overline{n}, \underline{n}) = \begin{cases} 0 & \text{if } \overline{n} \geq 1 \\ 1/\underline{n} & \text{if } \overline{n} = 0 \end{cases}, \quad \overline{q}_a(\overline{n}, \underline{n}) = \frac{1}{\overline{n}}.$$

A low type can only purchase if there is no high type present in the auction, because high types can always outbid low types in a bidding contest (see below). If indeed $\overline{n} = 0$ then each low type has an equal chance $1/\underline{n}$ of being served. High types, on the other hand, are not deterred by the presence of low types, which is why $\overline{q}_a = 1/\overline{n}$. For fixed price trading we have

$$\underline{q}_f(\overline{n}, \underline{n}) = \overline{q}_f(\overline{n}, \underline{n}) = \frac{1}{\overline{n} + \underline{n}},$$

which means that each customer, no matter what his type, has an equal chance of being served. Fixed price trading, unlike auctions, does not screen out customers ex-post; hence it is egalitarian at the point of service.

Sale Prices. With fixed price trading the sale price equals to the list price for all demand realizations, that is $p_{f,\overline{n},\underline{n}} = r$ for all $\overline{n}$ and $\underline{n}$. With second price auctions the reserve price r is charged if a single customer is present. In case multiple customers show up bidding ensues. We assume that a bid must be accompanied by a deposit of equal value. If the bidder wins, then he gets back the difference between the deposit and the sale price. Otherwise he gets back the entire deposit. This requirement effectively prevents buyers to bid over their budgets. Given that low types cannot overbid, it is straightforward to verify that the followings are dominant strategies: low types bid b and high types bid 1.[5]

Suppose $r \leq b$. Given the strategies, the sale price equals to

$$p_{a,\overline{n},\underline{n}} = \begin{cases} r & \text{if } \underline{n} + \overline{n} = 1 \\ b & \text{if } \overline{n} \leq 1 \text{ and } \underline{n} + \overline{n} \geq 2 \\ 1 & \text{if } \overline{n} \geq 2 \end{cases}.$$

If a single customer shows up, be it a high type or a low type, the reserve price is charged. If there are multiple customers and at most one of them is a high type then the sale price b. Finally if there

deposit is transferred to the seller; otherwise it is returned to its rightful owner at no cost. Such a practice prevents low types from showing up at unaffordable stores. A financial disclosure requirement is also effective.

[5] The picture changes if low types are allowed to overbid. Indeed if all low types bid just b then one of them can win the item for sure by bidding $b+\varepsilon$ (assuming no high types are present at the store). So the pair of strategies where low types bid b and high types bid 1 no longer constitute an equilibrium. In this case, one needs to specify in the auction rules what happens when the winner is in default; for instance one can assume that he does not receive the item plus he incurs some dis-utility c (because of punishment, legal consequences etc.). Going with this interpretation one can show that in an auction where only low types are present, say n of them, there exists a unique symmetric mixed strategy equilibrium where each buyer overbids with some positive probability. In particular this probability falls with n and c, i.e. buyers are less likely to overbid if the punishment is high or if they face stiffer competition. Yet the probability is non-zero for any finite value of c, i.e. no matter how large the punishment is, it does not prevent buyers from overbidding. In the full fledged model, given the respective arrival rates of high and low types, each customer needs to figure out how many of the n customers at a store are high/low types. Given the posterior probabilities, mixed strategies then can be obtained. The outcome, however, is non-trivial. To detour this complications, we simply assume that each bid must be accompanied by a deposit of equal value. See Zheng [15] for an auction setting where default is an option.

are multiple high types, then the the sale price is 1. If $r > b$ then $\underline{n} = 0$ as low types cannot afford to visit. So,

$$p_{a,\overline{n},\underline{n}} = \begin{cases} r & \text{if } \overline{n} = 1 \\ 1 & \text{if } \overline{n} \geq 2 \end{cases}.$$

Buyers. Buyers observe sellers' selections and choose to visit a store. The expected utilities are given by

$$\begin{aligned} U_H\left(r, \overline{\lambda}_m, \underline{\lambda}_m | m\right) &= \sum_{\overline{n}=0} \sum_{\underline{n}=0} z_{\overline{n}}\left(\overline{\lambda}_m\right) z_{\underline{n}}\left(\underline{\lambda}_m\right) \overline{q}_m\left(\overline{n}+1, \underline{n}\right)\left(1 - p_{\overline{n}+1,\underline{n},m}\right), \\ U_L\left(r, \overline{\lambda}_m, \underline{\lambda}_m | m\right) &= \mathbb{I}(b) \sum_{\overline{n}=0} \sum_{\underline{n}=0} z_{\overline{n}}\left(\overline{\lambda}_m\right) z_{\underline{n}}\left(\underline{\lambda}_m\right) \underline{q}_m\left(\overline{n}, \underline{n}+1\right)\left(1 - p_{\overline{n},\underline{n}+1,m}\right), \end{aligned}$$

where $\mathbb{I}(b)$ is an indicator function satisfying $\mathbb{I}(b) = 0$ if $r > b$ and $\mathbb{I}(b) = 1$ if $r \leq b$. A quick interpretation of the first line is this. With probability $z_{\overline{n}} z_{\underline{n}}$ a high type buyer who arrives at a store encounters $\overline{n}$ high type and $\underline{n}$ low type competitor buyers. He purchases with probability $\overline{q}_m(\overline{n}+1, \underline{n})$ and his payoff is $1 - p_{\overline{n}+1,\underline{n},m}$. The second line is interpreted similarly except one needs to account for affordability.

Lemma 2.1 *We have*

$$\frac{\partial U_H(\cdot|m)}{\partial r} < 0, \ \frac{\partial U_H(\cdot|m)}{\partial \overline{\lambda}_m} < 0, \ \text{and } \frac{\partial U_H(\cdot|m)}{\partial \underline{\lambda}_m} < 0 \text{ for } m = a, f.$$

Assuming $r \leq b$, the partial derivatives of U_L with respect to r, $\overline{\lambda}_m$ and $\underline{\lambda}_m$ have the same signs.

Proof. See the proof of Lemma 1 in [3].

Put simply, the Lemma says that buyers dislike expensive and crowded stores. The sign of the first partial derivative is obvious. For the second and the third note that a larger $\overline{\lambda}_m$ or $\underline{\lambda}_m$ shifts the probability mass from low to high demand realizations. Such a shift causes the expected utility to decline because customer are less likely to be served at stores with high demand realizations.

A buyer visits a store only if his expected payoff from visiting there is as high as he would obtain anywhere else. More precisely $\overline{\lambda}_m$ and $\underline{\lambda}_m$ must satisfy

$$\overline{\lambda}_m \begin{cases} = 0 & \text{if } \ U_H < \overline{U} \\ \in (0, \infty) & \text{if } \ U_H = \overline{U} \end{cases} \quad \text{and} \quad \underline{\lambda}_m \begin{cases} = 0 & \text{if } \ U_L < \underline{U} \\ \in (0, \infty) & \text{if } \ U_L = \underline{U} \end{cases}, \tag{3}$$

where

$$\overline{U} := \max_{r \in [0,1], m \in \mathbb{M}} U_H\left(r, \overline{\lambda}_m, \underline{\lambda}_m | m\right) \text{ and } \underline{U} := \max_{r \in [0,1], m \in \mathbb{M}} U_L\left(r, \overline{\lambda}_m, \underline{\lambda}_m | m\right).$$

We refer to $\overline{U}$ and $\underline{U}$ as the market utilities of high and low type customers. Lemma 2.1 implies that buyers can be indifferent to stores by adjusting the queue lengths. Indeed they may show up at stores that offer unattractive terms of trade as long as they expect fewer competitors there. Conditions (2) and (3) uniquely pin down the queue lengths across stores.

Because of strong symmetry the indifference condition (3) must hold on *and* off the equilibrium path, i.e. buyers out of equilibrium must behave identically and must be indifferent across sellers.[6] So, the out-of-equilibrium distribution of demand at a deviant store is still Poisson, where queue lengths $\overline{\lambda}'_m$ and $\underline{\lambda}'_m$ satisfy (3). Notice, however, market utilities are not affected by a deviation.

[6] See Abreu [1].

The reason is that in a large economy the covariance of demand across stores vanishes; hence a change in the probability of visiting a particular store does not affect the distribution of demand at other stores (see [2], [10], [11]).

Sellers. The expected profit of a seller depends on the sale mechanism in place m, the reserve price r and the queue lengths $\overline{\lambda}_m, \underline{\lambda}_m$. We have

$$\Pi\left(r, \overline{\lambda}_m, \underline{\lambda}_m | m\right) = \sum_{\overline{n}=0} \sum_{\underline{n}=0} z_{\overline{n}}\left(\overline{\lambda}_m\right) z_{\underline{n}}\left(\underline{\lambda}_m\right) p_{\overline{n},\underline{n},m}.$$

With probability $z_{\overline{n}} z_{\underline{n}}$ the seller meets $\overline{n}$ high type and $\underline{n}$ low type customers and sells at the price $p_{\overline{n},\underline{n},m}$; his payoff is zero if he does not get a customer. The problem of a of a seller is given by

$$\max_{m \in \mathbb{M},\ r \in [0,1],\ \left(\overline{\lambda}_m, \underline{\lambda}_m\right) \in \mathbb{R}_+^2} \Pi\left(r, \overline{\lambda}_m, \underline{\lambda}_m | m\right) \quad \text{subject to (3).} \tag{4}$$

Indifference conditions in (3) determine the queue lengths $\overline{\lambda}_m, \underline{\lambda}_m$ as functions of the trading mechanism m and the list price r. In particular $\overline{\lambda}_m$ and $\underline{\lambda}_m$fall in r[7] which means that the seller faces a trade off between revenue (intensive margin) and expected demand (extensive margin): on the one hand there is the desire to sell at a high price but on the other hand there is the fear of not being able to trade.

Definition 2.1 *A strongly symmetric subgame perfect equilibrium consists of a tuple $\left(\overline{\lambda}, \underline{\lambda}, \mathbf{r}\right)$satisfying the demand distribution (1), indifference (3) and profit maximization (4).*

The fraction of stores trading under each mechanism, given by (2) and also implicitly part of the equilibrium, can easily be recovered from the conditions above.

3 Analysis

3.1 Homogeneous Buyers

To build intuition consider the case with homogeneous buyers (no budget constraints).

Lemma 3.1 *Suppose that buyers are homogeneous. There exists a continuum of equilibria where either mechanism may be offered by any fraction of sellers. Sellers competing via fixed price trading post*

$$r_f(\lambda) = 1 - \frac{z_1(\lambda)}{1 - z_0(\lambda)} \tag{5}$$

while sellers trading via auctions post $r_a(\lambda) = 0$. Mechanisms are payoff equivalent; either mechanism delivers an expected profit of

$$\pi(\lambda) = 1 - z_0(\lambda) - z_1(\lambda). \tag{6}$$

Buyers randomize across stores and expect to earn $z_0(\lambda)$.

[7] Suppose $r \le b$. Observe that with fixed pricing $U_H(\cdot|f) = U_L(\cdot|f)$, which means that the constraints $U_H = \overline{U}$ and $U_L = \underline{U}$ are identical. Applying the Implicit Function Theorem one obtains

$$\frac{dr}{d\overline{\lambda}_f} = -\frac{\partial U_H/\partial \overline{\lambda}_f}{\partial U_H/\partial r} < 0 \text{ and } \frac{dr}{d\underline{\lambda}_f} = -\frac{\partial U_H/\partial \underline{\lambda}_f}{\partial U_H/\partial r} < 0.$$

The numerators and the denominators are both negative from Lemma 2.1. Similarly one can show that if $r > b$ then $dr/d\overline{\lambda}_f < 0$.

With auctions we have $U_H(\cdot|a) > U_L(\cdot|a)$, so, one needs the General Implicit Function Theorem to show that $\overline{\lambda}_a$ and $\underline{\lambda}_a$ fall in r. In the proof of Proposition 3.2 we establish these relationships.

The proof is in the appendix. The payoff equivalence result is in line with Kultti [7] and Eeckhout and Kircher [5]; see also Camera and Selcuk [3]. The fact that $r_f > r_a$ reveals why different mechanisms such as price posting and auctions may coexist in equilibrium. Sellers trading via auctions correctly anticipate that they will end up charging more than the reserve price, so they set $r_a = 0$ which clearly is below what fixed price traders ask. On expected terms, however, all sellers earn the same.

3.2 Ex-ante Screening: Fixed Price Trading

The following proposition outlines possible outcomes in the full fledged model when sellers compete via price posting.

Proposition 3.1 *Suppose $\mathbb{M}$ consists of fixed pricing only. Depending on the severity of the budget constraint there are three possible outcomes.*

1. Interior Equilibrium: If $r_f(\lambda) < b$ then the budget constraint is slack; the equilibrium with homogeneous buyers remains.

2. Corner Equilibrium: If $b < \pi\left(\overline{\lambda}\right)$ then all sellers advertise $r_f\left(\overline{\lambda}\right) > b$; so, only high budget customers can afford to shop. Low types are screened out completely.

3. Separating Equilibrium: If

$$\pi\left(\overline{\lambda}\right) \leq b \leq r_f(\lambda) \tag{7}$$

then a fraction φ of stores ('cheap stores') post $r_C = b$ while remaining stores ('expensive stores') post $r_E > b$. Low types can only afford to shop at cheap stores whereas high types strictly prefer shopping at expensive stores i.e. they avoid shopping at cheap stores.

Figure 1a provides an illustration (both figures are drawn for $\lambda = 1$). The interior equilibrium exists if b is high; the corner equilibrium exists if b is low and the separating equilibrium exists if b is moderate.

Proof. The first claim is obvious. For the second, conjecture an outcome where sellers target high types only. This means that the effective buyer-seller ratio shrinks to $\overline{\lambda}$; hence sellers post $r_f\left(\overline{\lambda}\right)$ and consequently earn $\pi\left(\overline{\lambda}\right)$. Now consider a seller who unilaterally deviates by posting some $r' \leq b$. He gets all low types for sure and therefore earns b for sure. But since he trades via fixed pricing he cannot earn more than b. The condition $b < \pi\left(\overline{\lambda}\right)$ guarantees that there is no profitable deviation.

For the third claim suppose (7) holds. In this parameter region conjecture an outcome where a fraction φ of stores (cheap stores) advertise $r_C = b$ targeting low types while the remaining stores (expensive stores) advertise some $r_E > b$ targeting high types. We further conjecture that high types strictly prefer expensive stores and do not shop at cheap stores (to be verified later). Along the conjecture the queue length at a cheap store equals to $\underline{\lambda}/\varphi := \lambda_C$. It follows that the expected earnings for sellers and buyers are

$$\Pi_C := [1 - z_0(\lambda_C)]\, b \quad \text{and} \quad U_C := \frac{1 - z_0(\lambda_C)}{\lambda_C}(1 - b).$$

Expensive stores receive high types only; so the queue length equals to $\lambda_E := \overline{\lambda}/(1-\varphi)$. The problem of an expensive store is similar to the one analyzed in the homogeneous model, except the effective buyer-seller ratio is λ_E instead of λ, hence we have

$$r_E := r_f(\lambda_E); \quad \Pi_E := \pi(\lambda_E); \quad U_E := z_0(\lambda_E).$$

The value of φ is pinned down by the equal profit condition $\Pi_E = \Pi_C$. Let

$$\Delta(\varphi) = \Pi_E - \Pi_C = \pi(\lambda_E) - [1 - z_0(\lambda_C)]\, b$$

and note that (i) $d\Delta/d\varphi > 0$, (ii) $\Delta(1) = 1 - \left(1 - e^{-\bar{\lambda}}\right) b > 0$, and (iii) $\Delta(0) = \pi\left(\bar{\lambda}\right) - b < 0$, where (i) and (ii) are obvious and (iii) follows from condition (7). The Intermediate Value Theorem implies that there exists some $\varphi \in (0,1)$ satisfying the equal profit condition. Now we need to verify the preceding conjectures; namely we need (i) $r_E > b$ so that expensive stores are unaffordable and (ii) $U_E > U_C$ so that high types deem cheap stores inferior. The following Lemma establishes this task. ■

Lemma 3.2 *We have: (i) $\varphi < l$ and therefore $\lambda_E < \lambda < \lambda_C$, (ii) $r_E > b$, and (iii) $U_E > U_C$.*

Proof. Start by showing that $\varphi < l$. By contradiction suppose that $\varphi = l$. This means that $\lambda_E = \lambda_C = \lambda$ and therefore

$$\Delta = \pi(\lambda) - [1 - z_0(\lambda)]\, b = [1 - z_0(\lambda)]\, [r_f(\lambda) - b]\,.$$

By condition (7) the expression $r_f(\lambda) - b$ is positive, which implies that $\Delta > 0$ contradicting the equilibrium condition $\Delta = 0$. Hence $\varphi \neq l$. The inequality gets worse when $\varphi > l$ because Δ rises in φ. Hence, the only possibility is $\varphi < l$, and therefore we have $\lambda_E < \lambda < \lambda_C$. Next we show that $r_E > b$. Solving the equilibrium condition $\Delta = 0$ for b one obtains

$$b = r_E \times \frac{1 - z_0(\lambda_E)}{1 - z_0(\lambda_C)}.$$

Since $\lambda_E < \lambda < \lambda_C$ the result follows. Finally we show that $U_E > U_C$. We have

$$\begin{aligned} U_E - U_C &= z_0(\lambda_E) - \frac{1 - z_0(\lambda_C) - \Pi_C}{\lambda_C} \\ &= z_0(\lambda_E) - \frac{z_0(\lambda_E) + z_1(\lambda_E) - z_0(\lambda_C)}{\lambda_C} \\ &= \frac{z_0(\lambda_C)}{\lambda_C}\, [e^x(x-1) + 1] > 0 \end{aligned}$$

where $x = \lambda_C - \lambda_E > 0$. In the first line we used (16), in the second line we substituted $\Pi_{E.}$ for Π_C and finally, in the third line note that the expression $e^x(x-1)+1$ is positive for all $x \neq 0$. ■

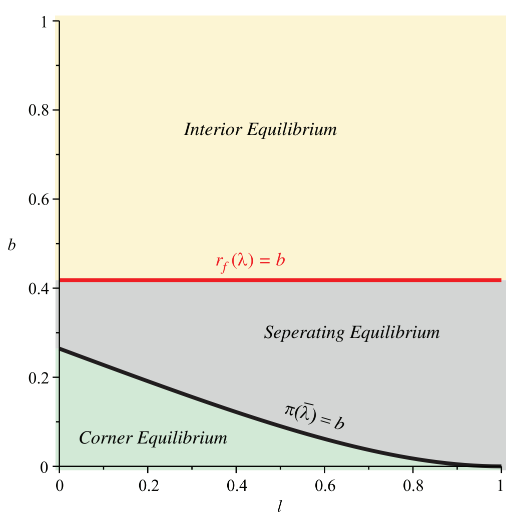

FIGURE 1A
Outcomes with Fixed Price Trading

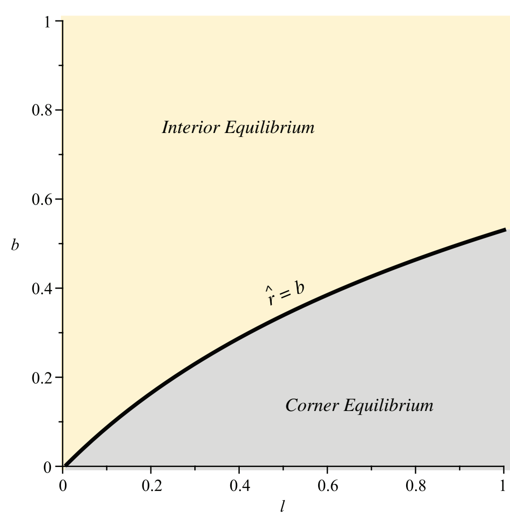

FIGURE 1B
Outcomes with Auctions

Lemma 3.2 tells that cheap stores are more crowded than expensive stores. So, the trade risk—the risk of not being able to purchase—is greater at cheap stores. This is why high type customers avoid shopping at such stores. In addition observe that implicit behind these results is the use of cash bonds to implement ex-ante screening.

3.3 Ex-post Screening: Second Price Auctions

Now, suppose that sellers trade via second price auctions only. We focus on outcomes where all sellers advertise the same reserve price $r \leq b$ and buyers randomize across stores (Proposition 3.3 demonstrates that no equilibrium exists with $r > b$). Along such an equilibrium path the queue of a seller consists of $\underline{\lambda}$ low types plus $\overline{\lambda}$ high types. Expected utilities are

$$U_H\left(r,\overline{\lambda},\underline{\lambda}\right) = \overline{z}_0\underline{z}_0\,(1-r) + \overline{z}_0\,(1-\underline{z}_0)\,(1-b) \text{ and} \tag{8}$$

$$U_L\left(r,\overline{\lambda},\underline{\lambda}\right) = \overline{z}_0\underline{z}_0\,(1-r) + \overline{z}_0\frac{1-\underline{z}_0-\underline{z}_1}{\underline{\lambda}}\,(1-b)\,, \tag{9}$$

where $\overline{z}_n := z_{\overline{n}}\left(\overline{\lambda}\right)$ and $\underline{z}_n$ is likewise.[8] The expected profit of a seller is given by

$$\Pi\left(r,\overline{\lambda},\underline{\lambda}\right) = \left(\overline{z}_0\underline{z}_1 + \overline{z}_1\underline{z}_0\right)r + \left[\overline{z}_0\left(1-\underline{z}_0-\underline{z}_1\right) + \overline{z}_1\left(1-\underline{z}_0\right)\right]b + 1 - \overline{z}_0 - \overline{z}_1.$$

With probability $\overline{z}_0\underline{z}_1 + \overline{z}_1\underline{z}_0$ the seller gets exactly one customer and charges the reserve price r. The expression in square brackets is the probability of getting no high types and at least two low types plus the probability of getting exactly one high type and at least one low type. In either case the sale price equals to b. Finally $1-\overline{z}_0-\overline{z}_1$ is the probability of getting multiple high types so the sale price is 1. Using (8), (9) and rearranging Π we obtain

$$\Pi\left(r,\overline{\lambda},\underline{\lambda}\right) = 1 - \overline{z}_0\underline{z}_0 - \overline{\lambda}U_H - \underline{\lambda}U_L. \tag{10}$$

The expression $1-\overline{z}_0\underline{z}_0$ can be interpreted as the expected revenue. It is the value created by a sale (one), multiplied by the probability of trading $1-\overline{z}_0\underline{z}_0$. Given $(\overline{\lambda},\underline{\lambda},r)$ one can interpret $\overline{\lambda}U_H + \underline{\lambda}U_L$ as the expected cost. The seller promises payoff U_H to each high type and U_L to each low type customer; since the queue has $\overline{\lambda}$ high types and $\underline{\lambda}$ low types the expected total cost equals to $\overline{\lambda}U_H + \underline{\lambda}U_L$. Observe that (10) collapses to (16) when $b=1$ and $l=0$ (homogeneous buyers).

The seller's problem is $\max_{r\in[0,b],\left(\overline{\lambda},\underline{\lambda}\right)\in\mathbb{R}^2_+} \Pi\left(r,\overline{\lambda},\underline{\lambda}\right)$ subject to

$$U_H\left(r,\overline{\lambda},\underline{\lambda}\right) \leq \overline{U} \text{ with equality if } \overline{\lambda} > 0$$
$$U_L\left(r,\overline{\lambda},\underline{\lambda}\right) \leq \underline{U} \text{ with equality if } \underline{\lambda} > 0,$$

where $\overline{U}$ and $\underline{U}$, which are taken parametrically, are market utilities for high and low types.

[8] U_H is interpreted as follows. With probability $\overline{z}_0\underline{z}_0$ the high type buyer is alone at the store, so he pays the reserve price r and obtains the item. With probability $\overline{z}_0\left(1-\underline{z}_0\right)$ some low types are present yet he is the only high type; so he overbids them and buys the good for sure paying b. With the complementary probability he finds other high types at the store in which case he ends up with zero surplus whether he acquires the item or not. U_L can be interpreted similarly. With probability $\overline{z}_0\underline{z}_0$ the low type buyer is alone at the store; so, he pays the reserve price r and gets the item for sure. If he encounters other low types, but not a high type, then the sale price is b and the good is allocated randomly. The expression in front of $1-b$ is the probability of finding no high types multiplied by the probability of finding at least another low type *and* acquiring the item. Indeed $\sum_{\underline{n}=1}^{\infty}\frac{z_{\underline{n}}(\underline{\lambda})}{\underline{n}+1} = \frac{1}{\underline{\lambda}}\sum_{\underline{n}=2}^{\infty} z_{\underline{n}}\left(\underline{\lambda}\right) = \frac{1-z_0(\underline{\lambda})-z_1(\underline{\lambda})}{\underline{\lambda}}$. Finally, if a high type is present then a low type has no chance of buying the good.

Lemma 3.3 *Suppose* $\mathbb{M}$ *includes second price auctions only. An outcome where some sellers advertise* $r > b$ *cannot be sustained as an equilibrium.*

Proof. By contradiction suppose there exists an equilibrium where some stores (expensive stores) advertise $r^* > b$. Their queue $\overline{\lambda}^*$ consists of high types only. Let $\overline{z}_0^* := z_0\left(\overline{\lambda}^*\right)$ and note that the expected utility of a high type visiting an expensive store equals to $U_H^* = \overline{z}_0^*\left(1 - r^*\right)$, whereas the expected profit of such a store is $\Pi^* = 1 - \overline{z}_0^* - \overline{\lambda}^* U_H^*$. Below we demonstrate that if a particular expensive seller (the deviant seller) posts b instead of r^* then he can do better than Π^* while providing high types exactly with payoff U_H^*. Hence the outcome cannot correspond to an equilibrium.

Since the deviant seller posts b, he gets low types as well as high types, so let $\underline{\lambda}'$ and $\overline{\lambda}'$ denote his expected demand consisting of low and high types, respectively. Substitute $r = b$ into (8) and (9) to obtain

$$U_H' = \overline{z}_0'\left(1 - b\right) \quad \text{and} \quad U_L' = \overline{z}_0' \frac{1 - \underline{z}_0'}{\underline{\lambda}'}\left(1 - b\right),$$

where $\overline{z}_n' := z_n\left(\overline{\lambda}'\right)$ and $\underline{z}_n' := z_n\left(\underline{\lambda}'\right)$. Use (10) and the expressions for U_H' and U_L' to obtain the expected profit of the deviant seller:

$$\Pi' = 1 - \overline{z}_0' - \overline{z}_1' + b\overline{z}_1' + b\overline{z}_0'\left(1 - \underline{z}_0'\right).$$

Now we show that $\Pi' > \Pi^*$ when $U_H' = U_H^*$. Note that

$$U_H' = U_H^* \Leftrightarrow \overline{z}_0'\left(1 - b\right) = \overline{z}_0^*\left(1 - r^*\right),$$

implying that $\overline{\lambda}' > \overline{\lambda}^*$ since $r^* > b$. The last term in Π' is positive hence

$$\Delta = \overline{z}_0^* - \overline{z}_0' + \overline{z}_1'(1 - r^*) - \overline{z}_1'\left(1 - b\right) > 0 \Rightarrow \Pi' > \Pi^*.$$

Substitute $\overline{z}_0'\left(1 - b\right)$ for $\overline{z}_0^*\left(1 - r^*\right)$ into Δ and rearrange to obtain

$$\Delta = \overline{z}_0'\left[e^x - 1 - \left(1 - b\right)x\right],$$

where $x = \overline{\lambda}' - \overline{\lambda}^* > 0$. The expression inside the square brackets is positive for all $x > 0$, hence $\Delta > 0$, and therefore $\Pi' > \Pi^*$. ■

The Lemma rules out a potential pooling equilibrium, where all sellers advertise $r > b$, as well as a separating equilibrium, where a fraction of sellers advertise $r > b$ while the rest advertise b or less. So with second price auctions—unlike fixed pricing—low types are never screened out ex-ante; neither partially nor completely. This is true no matter how small the budget is or how few the low types are. The intuition is this. Being expensive and catering to high types pays off if the expensive store possesses some distinct advantage. With fixed price trading there is such an advantage: expensive stores are less crowded (low types cannot afford to show up there) hence high types are more likely to be served. But when the trading mechanism is an auction then high types are not deterred by the presence of low types; they can outbid them. The advantage disappears hence the outcome with $r > b$ cannot be sustained as an equilibrium.

If an auction equilibrium exists it must be with all sellers catering to both types of customers. The following proposition describes two such outcomes; one with $r < b$ and the other with $r = b$.

Proposition 3.2 *Suppose* $\mathbb{M}$ *includes second price auctions only. There are two possible outcomes:*

1. Interior Equilibrium: If

$$\widehat{r} := \frac{\underline{\lambda} - \underline{z}_1 - \underline{\lambda} \underline{z}_1}{\underline{z}_0 - \underline{z}_0^2 + \underline{\lambda} \underline{z}_1 - \underline{z}_1} (1 - b) \leq b \tag{11}$$

then all sellers post $r = \widehat{r}$.

2. Corner Equilibrium: If $\widehat{r} > b$ *then all sellers post* $r = b$.

The proof is in the appendix. Figure 1b provides an illustration: if b is large and/or if l is small then $\widehat{r}$ is below b, so the interior equilibrium emerges; otherwise the corner equilibrium emerges.

Observe that $\widehat{r}$ is an explicit function of the parameters λ, b and l. In particular if $b = 1$ or $l = 0$ (i.e. no budget constraints) then $\widehat{r} = 0$ which is the equilibrium reserve price of auctions in a homogeneous setting (see Lemma 3.1). One can verify that $d\widehat{r}/db < 0$ and $d\widehat{r}/dl > 0$ i.e. sellers raise $\widehat{r}$ in response to a drop in b or an increase in l. Basically sellers offset the shortfall in profits due to budget constraints by raising the reserve price.

Finally note that, be it the corner case or the interior case, reserve prices are affordable, so all sellers expect to have a queue consisting of $\overline{\lambda} + \underline{\lambda} = \lambda$.

3.4 Efficiency

Proposition 3.3 *Suppose* $\mathbb{M}$ *includes fixed pricing and second price auctions. An outcome where all sellers compete in second price auctions is constrained efficient, whereas an outcome where all sellers trade via fixed pricing is inefficient except the interior case (i.e. except when* $r_f(\lambda) \leq b$*).*

Proof. Recall that low and high type buyers are identical in terms of their valuation of the good. If trade occurs at some price p, the buyer, no matter what his type, obtains payoff $1 - p$; hence the total surplus equals to 1. Therefore, as in the case with homogeneous buyers, efficiency is is synonymous with maximizing the the total number of trades in the market. The proof of the proposition amounts to showing that an auction outcome yields strictly more trades than a fixed price outcome. In an auction outcome, be it the the corner case or the interior case, the queue length of a seller equals to $\overline{\lambda} + \underline{\lambda} = \lambda$. Therefore the probability of trade at a given store is $1 - e^{-\lambda}$. With fixed price trading there are three cases. (i) Interior equilibrium: the budget constraint is slack hence sellers receive both types of buyers. This is identical to the case with auctions. (ii) Corner equilibrium: sellers ignore low types and trade with high types only. The probability of trade equals to $1 - e^{-\overline{\lambda}}$. Since $\lambda > \overline{\lambda}$ this outcome has less trade than an auction outcome, hence it is inefficient. (iii) Separating equilibrium: expensive stores trade with high types and cheap stores trade with low types. The weighted probability of trade equals to

$$1 - \varphi e^{-\lambda_C} - (1 - \varphi) e^{-\lambda_E}.$$

An auction outcome yields more trades if

$$\varphi e^{-\lambda_C} + (1 - \varphi) e^{-\lambda_E} > e^{-\lambda}.$$

Recall that (i) $\varphi \lambda_C + (1 - \varphi) \lambda_E = \lambda$ and (ii) $\lambda_C > \lambda > \lambda_E$ (Lemma 3.2). Since e^{-x} is convex the inequality holds; so, the separating equilibrium outcome is also inefficient. ■

4 Auctions or Price Posting?

So far we have focused on outcomes where all sellers trade either via fixed pricing or auctions. Below we discuss what happens when sellers are free to choose. A first result is that an auction equilibrium fails to exist if b is large, i.e. if low types differ only slightly from high types.

Proposition 4.1 *Suppose $\mathbb{M}$ includes fixed pricing and second price auctions. If*

$$b > \max\left\{\widehat{r}, 1 + \frac{U_L\left(\widehat{r}\right)\ln U_L\left(\widehat{r}\right)}{1 - U_L\left(\widehat{r}\right)} := \Psi_1\right\} \tag{12}$$

then there is no equilibrium where all sellers trade via second price auctions.

Proof. In the parameter space characterized by (12) we have $\widehat{r} < b$; so Proposition 3.2 implies that if an auction equilibrium exists it must be interior where sellers post $\widehat{r}$. In such an outcome agents' earnings are $U_H\left(\widehat{r}, \overline{\lambda}, \underline{\lambda}\right)$, $U_L\left(\widehat{r}, \overline{\lambda}, \underline{\lambda}\right)$ and $\Pi\left(\widehat{r}, \overline{\lambda}, \underline{\lambda}\right)$. Notice that $U_L < \overline{z}_0\underline{z}_0 < U_H$. Indeed

$$\begin{aligned} U_H > \overline{z}_0\underline{z}_0 &\Leftrightarrow \frac{1-\underline{z}_0}{\underline{z}_0}(1-b) > \widehat{r} \ \text{ and} \\ U_L < \overline{z}_0\underline{z}_0 &\Leftrightarrow \frac{1-\underline{z}_0-\underline{z}_1}{\underline{z}_1}(1-b) < \widehat{r}, \end{aligned}$$

both of which are true for all $\underline{\lambda}$, $\overline{\lambda}$ and b. These inequalities will prove useful below.

Now, consider a seller who switches to fixed pricing and posts some $r' < b$. Let U' denote the expected utility of buyers at the deviant store. Observe that U' is the same for both types of buyers since the seller competes via fixed pricing. Hence the deviant store attracts low budget customers only. To see why notice that

$$U' = \frac{1-\overline{z}_0'\underline{z}_0'}{\overline{\lambda}' + \underline{\lambda}'}\left(1-r'\right)$$

where $\overline{z}_0' := z_0\left(\overline{\lambda}'\right)$ and $\underline{z}_0' := z_0\left(\underline{\lambda}'\right)$. The queue lengths $\overline{\lambda}'$ and $\underline{\lambda}'$satisfy

$$\underline{\lambda}' \begin{cases} = 0 & \text{if } \ U' < U_L \\ \in (0,\infty) & \text{if } \ U' = U_L \\ = \infty & \text{if } \ U' > U_L \end{cases} \quad \text{and} \quad \overline{\lambda}' \begin{cases} = 0 & \text{if } \ U' < U_H \\ \in (0,\infty) & \text{if } \ U' = U_H \\ = \infty & \text{if } \ U' > U_H \end{cases}.$$

Since $U_H > U_L$, we have either $U' = U_L < U_H$ or $U_L < U_H = U'$. The latter is impossible since $U_L < U'$ implies that $\underline{\lambda} = \infty$, which in turn means that $U' = 0$ contradicting $U_H = U' > 0$. The former case, however, is feasible: $\underline{\lambda}' \in (0,\infty)$ adjusts to satisfy $U' = U_L$ whereas $\overline{\lambda}' = 0$ because $U' < U_H$.

Below we show that the deviant seller can provide low types the same utility U_L yet he can earn more than Π. His problem is

$$\max_{r'<b, \underline{\lambda}' \in \mathbb{R}_+} \left[1 - z_0\left(\underline{\lambda}'\right)\right] r' \ \text{ subj. to } \ U' = U_L$$

taking U_L as given. The FOC $z_0\left(\underline{\lambda}'\right) = U_L$ implies that he posts $r' = r_f\left(\underline{\lambda}'\right)$ and expects to earn

$$\Pi' = \pi\left(\underline{\lambda}'\right) = 1 - \underline{z}_0' - \underline{z}_1'.$$

The FOC further implies that $\underline{\lambda}' = -\ln U_L$; hence condition (12) guarantees that $r' < b$, which is what we have conjectured.

Below we verify that Π' exceeds Π. We have

$$\Pi' > \Pi \Leftrightarrow \overline{z}_0\underline{z}_0 + \underline{\lambda}U_L + \overline{\lambda}U_H - \underline{z}_0' - \underline{z}_1' > 0.$$

Since $U_H > U_L$ and $\overline{\lambda} + \underline{\lambda} = \lambda$ it suffices to show

$$\Delta := \overline{z}_0\underline{z}_0 + \lambda U_L - \underline{z}_0' - \underline{z}_1' > 0.$$

Observe that $\overline{z}_0\underline{z}_0 = e^{-\lambda}$. In addition the FOC $\underline{z}_0' = U_L$ implies that $\underline{\lambda}' > \lambda$ because $U_L < \overline{z}_0\underline{z}_0$. Substitute $\underline{z}_0' = U_L$ into Δ obtain

$$\Delta = \underline{z}_0' \left(e^x - 1 - x\right),$$

where $x := \underline{\lambda}' - \lambda > 0$. The expression inside the parentheses is positive for all $x > 0$. Hence the deviation is profitable. ■

The proposition establishes that if low types differ only slightly from high types then an auction equilibrium cannot survive the availability of fixed pricing. The intuition is this. With auctions low types are always second to high types at the point of service, no matter how small the budget difference is. A low type can only purchase if there is no high type around; and even then he has to compete against other low types. So, if a seller switches to fixed price trading, which is egalitarian at the point of service, then he can attract low types almost for sure and improve his payoff. Of course there needs to be 'enough room' so that he can post a profit maximizing price, that is b needs to be large enough; hence Condition (12). Figure 2a, which is drawn for $\lambda = 1$, depicts the parameter space outlined by Condition (12). Notice that the auction equilibrium fails to exist even when there are very few low types ($l \approx 0$) who have sufficiently large budgets.

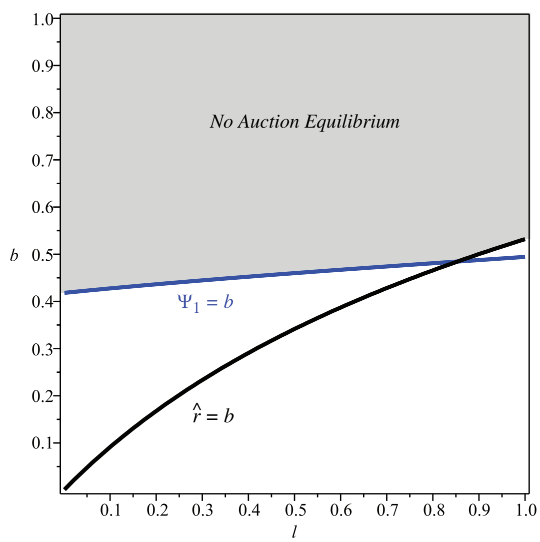

Figure 2a
Auction equilibrium fails to exist

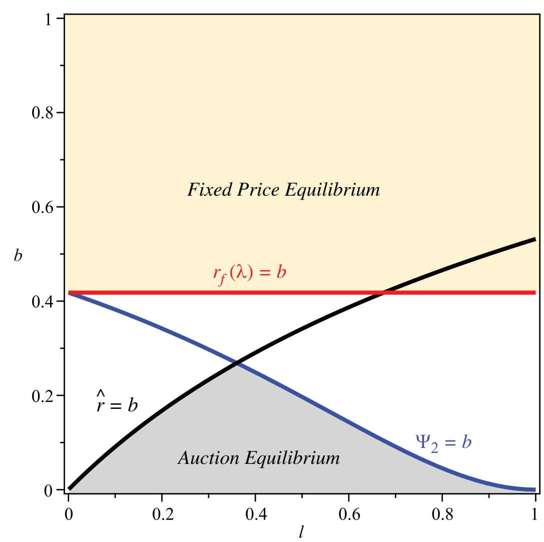

Figure 2b
Equilibria with $\mathbb{M}$={price posting, auctions}

In a similar setting McAfee [8] shows that the unique equilibrium entails all sellers holding second price auctions with a reserve and buyers randomizing across stores. Peters [9] generalizes McAfee's result with heterogeneous sellers. We demonstrate that existence of an auction equilibrium is not robust to budget constraints: merely lowering the budgets of a few buyers ($l \approx 0$) renders the auction equilibrium unsustainable.

The setup in McAfee [8] and others has buyers who differ in their valuations. In our model buyers have identical valuations, so Proposition 4.1 should apply in a heterogeneous setting as well. To illustrate this point, in Appendix II we extend the current setup by considering buyers who are heterogeneous in their valuations. In particular we let a fraction of customers have valuation $v < 1$ for the good and we assume that some of the high value customers have low budgets so that $b < v < 1$. We show that if buyers are sufficiently similar in terms of their valuations *and* budgets, i.e if condition (24) holds, then, again, the auction equilibrium cannot survive fixed price trading.

Finally we provide some sufficient conditions for the existence of a fixed price equilibrium and an auction equilibrium.

Proposition 4.2 . *Suppose $\mathbb{M}$ includes fixed pricing and second price auctions. If $r_f(\lambda) \leq b$ then all sellers trade via fixed pricing and post $r_f(\lambda)$. If*

$$b < \min\left\{\widehat{r}, \frac{1-\overline{z}_0-\overline{z}_1}{1-\overline{z}_0-\overline{z}_1+\overline{z}_0\underline{z}_0} := \Psi_2\right\} \tag{13}$$

then all sellers trade via auctions and post $r = b$. Both equilibria are constrained efficient.

The proof is in the appendix. The proposition outlines two areas in the parameter space where either fixed pricing or auctions emerge as the equilibrium trading mechanism; see Figure 2b for an illustration. The choice between price posting and second price auctions boils down to the degree of heterogeneity across customers, that is how different buyers are in terms of their ability to pay. If they differ only slightly, i.e. if $b \geq r_f(\lambda)$, then sellers prefer to trade via fixed pricing. If they differ significantly, i.e. if (13) holds, then sellers prefer second price auctions.

More precisely the proposition establishes two things. One, the interior fixed price equilibrium is robust to the availability of second price auctions. Two, the corner equilibrium with auctions is in general robust to availability of fixed pricing. The first result requires $r_f(\lambda) \leq b$ and it corresponds to a parameter space where b is large whereas the second result requires condition (13)[9] and so it covers a region where b is small. We have not made an attempt to characterize equilibria for intermediate values of b. Finally, constrained efficiency follows from the discussion in Section 3.4.

5 Concluding Remarks

The existing literature on competing mechanisms capitulates that if buyers differ in their valuations then the unique equilibrium entails all sellers holding second price auctions and if buyers are homogeneous then a large number of mechanisms are payoff equivalent, so sellers are indifferent. We show that these results are not robust to a fundamental source of heterogeneity: the presence of budget constrained buyers. If buyers differ only slightly in terms of their ability to pay then we show that sellers prefer fixed price trading; otherwise they prefer auctions.

Two observations are worth noting, however. First the model is built on the premise that the economy is large so that the the market utility assumption holds. In a finite economy, a deviation by an individual seller changes the outside option of of the buyers, so different outcomes may arise. Second, low budget customers are not allowed to overbid in the auction, which, again, simplifies the analysis but restricts the outcomes.

[9]In (13) $\widehat{r} > b$ is needed for the existence of the corner equilibrium and the expression after $\widehat{r}$ in the curly brackets is simply a sufficient condition that guarantees that a deviation to fixed pricing is indeed unprofitable.

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Appendix I: Omitted Proofs

Proof of Lemma 3.1. With homogeneous buyers we have $b = 1$ and $l = 0$ so that $\underline{\lambda}= 0$ and $\overline{\lambda} = \lambda$. The expected utility of a buyer becomes (buyers are identical so one can safely drop the type specific subscripts in the expression)

$$U(r, \lambda_m|m) = \sum_{n=0}^{\infty} z_n(\lambda_m) \frac{1 - p_{m,n+1}(r)}{n+1}$$
$$= \frac{1}{\lambda_m} \sum_{n=1}^{\infty} z_n(\lambda_m) \left[1 - p_{m,n}(r)\right]. \tag{14}$$

The second line follows from the fact that $z_{n+1}(\lambda_m) = \lambda_m z_n(\lambda_m)/(n+1)$. The expected profit of a seller is also simplified; we have

$$\Pi(r, \lambda_m|m) = \sum_{n=1}^{\infty} z_n(\lambda_m) p_{m,n}(r) \tag{15}$$
$$= 1 - z_0(\lambda_m) - \lambda_m U(r, \lambda_m|m). \tag{16}$$

We use (14) to obtain the second line. The seller's problem is

$$\max_{\lambda_m \in \mathbb{R}_+} 1 - z_0(\lambda_m) - \lambda_m U(r, \lambda_m|m) \quad \text{s.t. } U(r, \lambda_m|m) = \overline{U},$$

where $\overline{U}$ is the "market utility" for a buyer. The objective function is concave therefore, the first order condition

$$z_0(\lambda_m) = \overline{U}$$

corresponds to a maximum. Conjecture that the FOC holds for all $m \in \mathbb{M}$ (to be verified). This implies that

$$\overline{U} = z_0(\lambda_m) = z_0(\lambda_{m'}) \;\; \Rightarrow \;\; \lambda_m = \lambda_{m'}, \forall m, m' \in \mathbb{M}.$$

Insert $\lambda_m = \lambda_{m'}$ into (2) to obtain

$$\sum_{m \in \mathbb{M}} \gamma_m \lambda_m = \lambda \quad \Rightarrow \lambda_m = \lambda, \forall m,$$

which means that along the equilibrium path sellers have identical queue lengths. Use $\lambda_m = \lambda$ and (16) to rewrite the FOC as

$$\Pi(r, \lambda|m) = 1 - z_0(\lambda) - z_1(\lambda), \forall m. \tag{17}$$

Since $\mathbb{M}$ consists of fixed pricing and second price auctions, inserting the specific sale price functions into (15) to obtain

$$\Pi(r, \lambda|f) = r\left[1 - z_0(\lambda)\right] \quad \text{and} \quad \Pi(r, \lambda|a) = rz_0(\lambda) + 1 - z_0(\lambda) - z_1(\lambda).$$

Equilibrium list prices are obtained by solving (17) for r; we have

$$r_f = 1 - \frac{z_1}{1 - z_0} > r_a = 0.$$

Both prices satisfy the FOC with equality; verifying the earlier conjecture. It is immediate that under either mechanism the expected profit of sellers is $\pi(\lambda) = 1 - z_0(\lambda) - z_1(\lambda)$ while the expected utility to buyers is $z_0(\lambda)$. Hence in equilibrium sellers are indifferent between price posting and auctions. ■

Proof of Proposition 3.2. Lemma 3.3 establishes that if an auction equilibrium exists it must be such that all sellers cater to both types of customers. So the seller's problem is

$$\max_{r\in[0,b],\left(\overline{\lambda},\underline{\lambda}\right)\in\mathbb{R}_+^2} 1-\overline{z}_0\underline{z}_0-\overline{\lambda}U_H-\underline{\lambda}U_L$$

where U_H and U_L simultaneously satisfy the indifference conditions

$$U_H\left(r,\overline{\lambda},\underline{\lambda}\right)=\overline{U} \text{ and } U_L\left(r,\overline{\lambda},\underline{\lambda}\right)=\underline{U}.$$

The FOC is given by

$$\frac{d\Pi}{dr}=\left(\overline{z}_0\underline{z}_0-\overline{U}\right)\frac{d\overline{\lambda}}{dr}+\left(\overline{z}_0\underline{z}_0-\underline{U}\right)\frac{d\underline{\lambda}}{dr}=0. \tag{18}$$

The General Implicit Function Theorem implies that

$$\frac{d\overline{\lambda}}{dr}=\frac{\det\overline{B}}{\det A} \quad\text{and}\quad \frac{d\underline{\lambda}}{dr}=\frac{\det\underline{B}}{\det A},$$

where

$$A=\begin{bmatrix}\frac{\partial U_H}{\partial\overline{\lambda}} & \frac{\partial U_H}{\partial\underline{\lambda}}\\ \frac{\partial U_L}{\partial\overline{\lambda}} & \frac{\partial U_L}{\partial\underline{\lambda}}\end{bmatrix},\ \overline{B}=\begin{bmatrix}-\frac{\partial U_H}{\partial r} & \frac{\partial U_H}{\partial\underline{\lambda}}\\ -\frac{\partial U_L}{\partial r} & \frac{\partial U_L}{\partial\underline{\lambda}}\end{bmatrix},\ \underline{B}=\begin{bmatrix}\frac{\partial U_H}{\partial\overline{\lambda}} & -\frac{\partial U_H}{\partial r}\\ \frac{\partial U_L}{\partial\overline{\lambda}} & -\frac{\partial U_L}{\partial r}\end{bmatrix}.$$

Inspecting (8) and (9) one can verify that for $j=H,L$ we have

$$\begin{aligned}\frac{\partial U_j}{\partial r}&=-\overline{z}_0\underline{z}_0,\ \frac{\partial U_j}{\partial\overline{\lambda}}=-U_j,\ \frac{\partial U_H}{\partial\underline{\lambda}}=-\overline{z}_0\underline{z}_0\left(b-r\right)\\ \frac{\partial U_L}{\partial\underline{\lambda}}&=-\overline{z}_0\underline{z}_0\left(1-r\right)-\overline{z}_0\left(1-b\right)\frac{1-\underline{z}_0-\underline{z}_1-\underline{\lambda}\underline{z}_1}{\underline{\lambda}^2}.\end{aligned}$$

Observe that (i) $0<U_L<U_H$ and (ii) $\partial U_L/\partial\underline{\lambda}<\partial U_H/\partial\underline{\lambda}<0$. It follows that

$$\begin{aligned}\det A&=U_L\frac{\partial U_H}{\partial\underline{\lambda}}-U_H\frac{\partial U_L}{\partial\underline{\lambda}}>0,\\ \det\overline{B}&=\frac{\overline{z}_0^2\underline{z}_0}{\underline{\lambda}^2}\left(1-b\right)\left(-1+\underline{z}_0+\underline{z}_1\right)<0,\\ \det\underline{B}&=\frac{\overline{z}_0^2\underline{z}_0}{\underline{\lambda}}\left(1-b\right)\left(1-\underline{\lambda}-\underline{z}_0\right)<0.\end{aligned}$$

Substitute $U_H=\overline{U}$ and $U_L=\underline{U}$ into $d\Pi/dr$ to obtain

$$\frac{d\Pi}{dr}=\frac{\overline{z}_0}{\det A}\left(c_1r-c_2\right),$$

where

$$\begin{aligned}c_1&=\underline{z}_0\left(\det\overline{B}+\det\underline{B}\right)<0,\\ c_2&=\left(1-b\right)\left[\left(1-\underline{z}_0\right)\det\overline{B}+\frac{1-\underline{z}_0-\underline{z}_1}{\underline{\lambda}}\det\underline{B}\right]<0.\end{aligned}$$

Solving the FOC for r we obtain

$$\frac{d\Pi}{dr} = 0 \Leftrightarrow r = \frac{c_2}{c_1} = \frac{\underline{\lambda} - \underline{z}_1 - \underline{\lambda} \underline{z}_1}{\underline{z}_0 - \underline{z}_0^2 + \underline{\lambda} \underline{z}_1 - \underline{z}_1} (1-b) := \widehat{r}.$$

To verify the SOC note that $\overline{z}_0 / \det A > 0$ since $\det A > 0$. It follows that

$$sign\left(\frac{d\Pi}{dr}\right) = sign\left(c_1 r - c_2\right).$$

Observe that c_1 and c_2 are negative constants since $\det \overline{B}$ and $\det \underline{B}$ are both negative and independent of r. Therefore $d\Pi/dr > 0$ for all $r < \widehat{r}$ and $d\Pi/dr < 0$ for all $r > \widehat{r}$, which means that $r = \widehat{r}$ is the global maximum.

Recall that Π is defined under the conjecture $r \leq b$, so if $\widehat{r} \leq b$ then posting $r = \widehat{r}$ maximizes Π. Below we show that if $\widehat{r} > b$ then all sellers post $r = b$. First, observe that $\widehat{r} > b$ implies $\left.\frac{d\Pi}{dr}\right|_{r=b} > 0$, hence posting any $r < b$ is strictly inferior to posting $r = b$. Now suppose all stores post $r = b$ and buyers randomize across stores. In such an outcome agents' earnings are given by

$$\Pi = 1 - (1-b)\left(\overline{z}_0 + \overline{z}_1\right) - b\overline{z}_0\underline{z}_0,$$

$$U_H = \overline{z}_0 (1-b) \quad \text{and } U_L = \overline{z}_0 \frac{1-\underline{z}_0}{\underline{\lambda}} (1-b).$$

We will verify that there is no profitable deviation by posting $r' > b$. Indeed the deviant store attracts high types only, so his queue $\overline{\lambda}'$ satisfies

$$z_0\left(\overline{\lambda}'\right)\left(1-r'\right) = U_H,$$

which implies that

$$\overline{\lambda}' - \overline{\lambda} = \ln\left(1-r'\right) - \ln\left(1-b\right) < 0.$$

His expected profit equals to

$$\Pi' = z_1\left(\overline{\lambda}'\right) r' + 1 - z_0\left(\overline{\lambda}'\right) - z_1\left(\overline{\lambda}'\right).$$

Observe that

$$\begin{aligned} \Pi - \Pi' &= z_0\left(\overline{\lambda}'\right) + z_1\left(\overline{\lambda}'\right)\left(1-r'\right) - (1-b)\left(\overline{z}_0 + \overline{z}_1\right) - b\overline{z}_0\underline{z}_0 \\ &= \overline{z}_0 \frac{1-b}{1-r'} + \overline{z}_0 (1-b)\left(\overline{\lambda}' - \overline{\lambda} - 1\right) - b\overline{z}_0\underline{z}_0. \end{aligned}$$

It is straightforward to show that $\Pi > \Pi'$ if

$$\ln\left(1-r'\right) + \frac{1}{1-r'} > 1 + \frac{b\underline{z}_0}{1-b} + \ln\left(1-b\right).$$

The left hand side increases in r' so set $r' = b$ and observe that the inequality holds. ■

Proof of Proposition 4.2. To prove the first part of the proposition consider an interior price posting equilibrium where all sellers post $r_f(\lambda) \leq b$. The strategy is to show that this outcome remains as an equilibrium even if $\mathbb{M}$ includes second price auctions.

Along the said outcome the expected profit of sellers equals to $\pi(\lambda)$ and expected utility of buyers (for both types) equals to $z_0(\lambda)$. Now consider a seller who unilaterally deviates by trading with auctions and suppose that he posts some reserve price r'. There are two cases to consider: $r' \leq b$ and $r' > b$. Start with the first case. Let $\overline{\lambda}'$ and $\underline{\lambda}'$ be the queues at the deviant store consisting of high and low types. Furthermore let $\overline{z}'_n := z_n\left(\overline{\lambda}'\right)$ and $\underline{z}'_n := z_n\left(\underline{\lambda}'\right)$. We have

$$
\begin{aligned}
U'_H &= \overline{z}'_0 \underline{z}'_0 (1-r') + \overline{z}'_0 (1-\underline{z}'_0)(1-b) \\
U'_L &= \overline{z}'_0 \underline{z}'_0 (1-r') + \overline{z}'_0 \frac{1-\underline{z}'_0-\underline{z}'_1}{\underline{\lambda}'}(1-b) \\
\Pi' &= 1-\overline{z}'_0 \underline{z}'_0 - \overline{\lambda}' U'_H - \underline{\lambda}' U'_L .
\end{aligned}
$$

These expressions follow from (8), (9) and (10). Note that the deviant seller attracts high types only even though he posts an affordable reserve price. To see why observe that $\overline{\lambda}'$ and $\underline{\lambda}'$ satisfy

$$
\underline{\lambda}' \begin{cases} = 0 & \text{if } \ U'_L < z_0(\lambda) \\ \in (0,\infty) & \text{if } \ U'_L = z_0(\lambda) \\ = \infty & \text{if } \ U'_L > z_0(\lambda) \end{cases} \quad \text{and} \quad \overline{\lambda}' \begin{cases} = 0 & \text{if } \ U'_H < z_0(\lambda) \\ \in (0,\infty) & \text{if } \ U'_H = z_0(\lambda) \\ = \infty & \text{if } \ U'_H > z_0(\lambda) \end{cases},
$$

Since $U'_H > U'_L$ we either have $z_0(\lambda) = U'_H > U'_L$ or $U'_H > U'_L = z_0(\lambda)$. The latter case is impossible since $U'_H > z_0(\lambda)$ implies that $\overline{\lambda}' = \infty$, which in turn implies $U'_H = 0$, which is a contradiction. The former case is feasible: $z_0(\lambda) = U'_H > U'_L$ implies that $\underline{\lambda}' = 0$ and $\overline{\lambda}' > 0$ satisfies the indifference condition $z_0(\lambda) = U'_H$. Thus, the deviant seller attracts high types only; low types are strictly better off with fixed price trading. So we have

$$
U'_H = \overline{z}'_0 (1-r') \quad \text{ and } \quad \Pi' = 1-\overline{z}'_0 - \overline{\lambda}' U'_H = 1-\overline{z}'_0 - \overline{\lambda}' z_0(\lambda).
$$

The indifference condition

$$
z_0(\lambda) = U'_H = \overline{z}'_0 (1-r')
$$

implies that $\lambda > \overline{\lambda}'$. Observe that

$$
\pi(\lambda) - \Pi' = z_0(\lambda)\left(e^x - 1 - x\right) > 0
$$

where $x := \lambda - \overline{\lambda}' > 0$. The expression inside the parenthesis is positive for all $x > 0$, hence the deviation is not a profitable one.

Now consider the case where $r' > b$. The deviant seller, again, attracts high types only; but now because low types cannot afford to shop at his store. Obviously this is identical to the scenario above; hence there is no deviation by advertising $r' > b$ either. So the interior fixed pricing equilibrium survives auctions.

For the second part of the proposition, suppose condition (13) holds. Since $\widehat{r} > b$, if an auction equilibrium exists it must be the corner case where all sellers post $r = b$ (Proposition 3.2). We will demonstrate that this equilibrium survives the availability of fixed price trading. Expected earnings along the said equilibrium are given by

$$
\begin{gathered}
U_H = \overline{z}_0 (1-b), \, U_L = \overline{z}_0 (1-b) \frac{1-\underline{z}_0}{\underline{\lambda}} \\
\Pi = 1-(1-b)(\overline{z}_0 + \overline{z}_1) - b\overline{z}_0 \underline{z}_0 .
\end{gathered}
$$

These expressions are obtained from (8), (9) and (10) by substituting $r = b$. Below we check if there is a profitable deviation to fixed pricing. Again, there are two cases to consider: $r' \leq b$ and

$r' > b$. Start with the case $r' \leq b$. Since the deviator trades via fixed pricing he can at most earn b. So a sufficient condition for no deviation is

$$\Pi > b \Leftrightarrow \frac{1 - \overline{z}_0 - \overline{z}_1}{1 - \overline{z}_0 - \overline{z}_1 + \overline{z}_0 \underline{z}_0} > b.$$

The inequality is part of condition (13). Now consider the second case with $r' > b$. We show that such deviation is not profitable. Observe that the deviator can attract high types only. Let $\overline{\lambda}'$ denote his queue length and let $\overline{z}'_n = z_n \left(\overline{\lambda}'\right)$. His problem is

$$\max_{r' > b, \overline{\lambda}' \in \mathbb{R}_+} (1 - \overline{z}'_0)\, r' \quad \text{subj. to} \quad \frac{1 - \overline{z}'_0}{\overline{\lambda}'} (1 - r') = U_H,$$

taking U_H as given. The problem is standard by now, so the FOC requires $\overline{z}'_0 = U_H$. Substituting for U_H we have

$$\overline{z}'_0 = \overline{z}_0 (1 - b) \Leftrightarrow \overline{\lambda}' - \overline{\lambda} = -\ln(1 - b). \tag{19}$$

The deviant seller's expected profit equals to

$$\Pi' = \pi \left(\overline{\lambda}'\right) = 1 - \overline{z}'_0 - \overline{z}'_1.$$

Now we can compare Π and Π'. Observe that

$$\begin{aligned}
\Pi - \Pi' &= \overline{z}'_0 \left(1 + \overline{\lambda}'\right) - (1 - b)\, \overline{z}_0 \left(\overline{\lambda} + 1\right) - b \overline{z}_0 \underline{z}_0 \\
&= \overline{z}_0 (1 - b) \left(\overline{\lambda}' - \overline{\lambda}\right) - b \overline{z}_0 \underline{z}_0 \\
&= -\overline{z}_0 (1 - b) \ln(1 - b) - b \overline{z}_0 \underline{z}_0.
\end{aligned}$$

The second and third lines make use of (19). It follows that

$$\Pi > \Pi' \Leftrightarrow -\frac{1 - b}{b} \ln(1 - b) > \underline{z}_0.$$

Condition (13) has that $b < \widehat{r}$. In the proof of Proposition 4.1 we have established that $\widehat{r} < (1 - \underline{z}_0)(1 - b) / \underline{z}_0$. Merging these inequalities we have

$$b < \widehat{r} < \frac{1 - \underline{z}_0}{\underline{z}_0} (1 - b) \Rightarrow \underline{z}_0 < 1 - b.$$

It follows that $\Pi > \Pi'$ if

$$-\frac{1 - b}{b} \ln(1 - b) > 1 - b.$$

It is straightforward to verify that the inequality above is satisfied for all $b \in (0, 1)$; hence the deviation is not profitable. ■

Appendix II: Customers with different valuations

Suppose a fraction η of customers have valuation $v < 1$ for the good while the rest have valuation 1. Moreover a fraction l of the high valuation customers have low budgets. We assume that $b < v < 1$. The rest of the model remains unchanged.

Our objective is not to characterize all possible outcomes for the entire parameter space. Rather we focus on the region outlined by condition (24) and characterize the equilibrium where all stores trade via auctions and buyers randomize over where they visit. This is the outcome McAfee [8] and others focus on. Then we show that this outcome cannot survive fixed price trading; i.e. a seller can unilaterally do better than holding an auction if he chooses price posting.

To start, suppose $\mathbb{M}$ includes second price auctions only. The dominant bidding strategies are as follows: high value-high budget types bid 1, high value-low budget types bid b and low value types bid v. As before we assume that a bid must be accompanied by a deposit of equal value to prevent overbidding.

Let $\overline{n}$ and $\underline{n}$ denote the number of high and low budget types present at a store. Also let $\widehat{n}$ denote the number of low value types. Given the bidding strategies and the fact that $b < v < 1$, the sale price is given by

$$p_n(r) = \begin{cases} r & \text{if } \underline{n} + \overline{n} + \widehat{n} = 1 \\ b & \text{if } \overline{n} + \widehat{n} \leq 1 \text{ and } \underline{n} + \overline{n} + \widehat{n} \geq 2 \\ v & \text{if } \overline{n} \leq 1 \text{ and } \overline{n} + \widehat{n} \geq 2 \\ 1 & \text{if } \overline{n} \geq 2 \end{cases}.$$

If exactly one customer is present then the reserve price is charged. If $\overline{n} + \widehat{n} \leq 1$ and $\underline{n} + \overline{n} + \widehat{n} \geq 2$ then the sale price equals to b. Observe that low budget types can acquire the item only if $\overline{n} + \widehat{n} = 0$; indeed they will be outbid when $\overline{n} = 1$ or $\widehat{n} = 1$. If $\overline{n} \leq 1$ and $\overline{n} + \widehat{n} \geq 2$ then the sale price equals to v. In this case the presence of low budget types is immaterial; the good is acquired either by a high-budget type (if $\overline{n} = 1$) or a low value type (if $\overline{n} = 0$). Finally the sale price is driven up to 1 is multiple high budget types are present.

We conjecture that in the parameter space outlined by condition (24) the equilibrium reserve price is affordable, i.e. $r < b$ (to be verified). This means that all sellers cater to all types of customers. For notational convenience let $\widehat{\lambda} := \eta\lambda$, $\underline{\lambda} := l(1-\eta)\lambda$ and $\overline{\lambda} = (1-l)(1-\eta)\lambda$ denote the queue lengths. Also let $\widehat{z}_n := z_n\left(\widehat{\lambda}\right)$, $\overline{z}_n := z_n\left(\overline{\lambda}\right)$ and $\underline{z}_n := z_n(\underline{\lambda})$. On the equilibrium path the expected profit of a seller is given by

$$\begin{aligned} \Pi\left(r, \overline{\lambda}, \underline{\lambda}, \widehat{\lambda}\right) &= [\overline{z}_1\widehat{z}_0\underline{z}_0 + \overline{z}_0\widehat{z}_1\underline{z}_0 + \overline{z}_0\widehat{z}_0\underline{z}_1] \times r + [\overline{z}_0\widehat{z}_0(1-\underline{z}_0-\underline{z}_1) + (\overline{z}_1\widehat{z}_0 + \overline{z}_0\widehat{z}_1)(1-\underline{z}_0)] \times b \\ &\quad + [\overline{z}_0(1-\widehat{z}_0-\widehat{z}_1) + \overline{z}_1(1-\widehat{z}_0)] \times v + [1-\overline{z}_0-\overline{z}_1] \times 1. \end{aligned}$$

The expressions in square brackets in front of r, b, v and 1 are the probabilities where these prices are charged (see above). Given our conjecture that all sellers cater to all types of customers, the queue lengths $\overline{\lambda}$, $\underline{\lambda}$, and $\widehat{\lambda}$ are simultaneously determined by

$$U^H\left(r, \overline{\lambda}, \underline{\lambda}, \widehat{\lambda}\right) := \overline{z}_0\widehat{z}_0\underline{z}_0(1-r) + \overline{z}_0\widehat{z}_0(1-\underline{z}_0)(1-b) + \overline{z}_0(1-\widehat{z}_0)(1-v) = \overline{U} \tag{20}$$

$$U^V\left(r, \overline{\lambda}, \underline{\lambda}, \widehat{\lambda}\right) := \overline{z}_0\widehat{z}_0\underline{z}_0(v-r) + \overline{z}_0\widehat{z}_0(1-\underline{z}_0)(v-b) = \widehat{U} \tag{21}$$

$$U^L\left(r, \overline{\lambda}, \underline{\lambda}, \widehat{\lambda}\right) := \overline{z}_0\widehat{z}_0\underline{z}_0(1-r) + \overline{z}_0\widehat{z}_0\frac{1-\underline{z}_0-\underline{z}_1}{\underline{\lambda}}(1-b) = \underline{U} \tag{22}$$

These expressions are similar to their counterparts in Section 3.3 and can be interpreted similarly. Observe that $\overline{U}$, $\widehat{U}$ and $\underline{U}$ denote the market utilities, which are taken as given. One can show that

$$\Pi = 1 - \overline{z}_0 \left(1 - \widehat{z}_0\right)(1 - v) - \overline{z}_0 \widehat{z}_0 \underline{z}_0 - \overline{\lambda U} - \widehat{\lambda}\widehat{U} - \underline{\lambda U}. \tag{23}$$

Again 1 can be interpreted as the revenue generated from a sale; $\overline{z}_0 \left(1 - \widehat{z}_0\right)(1 - v)$ is the revenue loss due to encountering a low value type, and $\overline{z}_0 \widehat{z}_0 \underline{z}_0$ is the loss due to not getting a customer at all.

Proposition 5.1 *Suppose $\mathbb{M}$ includes second price auctions only. If*

$$b > \max\left\{r^\star, 1 + \frac{U^L\left(r^\star\right)\ln U_L\left(r^\star\right)}{1 - U_L\left(r^\star\right)}\right\} \quad \textit{and} \quad v > v^\star := \frac{1 - \underline{z}_0}{\underline{\lambda}}(1 - b) + b, \tag{24}$$

where

$$r^\star = \frac{\underline{\lambda}(1 - b)(v - b)\left(1 - \underline{z}_0 - \underline{z}_1\right)}{(1 - b)\left(\underline{z}_0 - \underline{z}_0^2 - \underline{z}_1\right) + (v - b)\,\underline{\lambda z}_1},$$

then all sellers post $r = r^\star < b$ and buyers randomize across stores.

Proof. The seller's problem is $\max_{r\in[0,1] \text{ and } \left(\overline{\lambda},\underline{\lambda},\widehat{\lambda}\right)\in\mathbb{R}^3_+} \Pi$ subject to (20), (21) and (22). The FOC is given by

$$\frac{d\Pi}{dr} = \left[\overline{z}_0\left(1 - \widehat{z}_0\right)(1 - v) + \overline{z}_0\widehat{z}_0\underline{z}_0 - \overline{U}\right]\frac{d\overline{\lambda}}{dr} + \left[\overline{z}_0\widehat{z}_0\underline{z}_0 - \underline{U}\right]\frac{d\underline{\lambda}}{dr} + \left[\overline{z}_0\widehat{z}_0\underline{z}_0 - \overline{z}_0\widehat{z}_0(1 - v) - \widehat{U}\right]\frac{d\widehat{\lambda}}{dr} = 0.$$

The General Implicit Function Theorem implies that

$$\frac{d\overline{\lambda}}{dr} = \frac{\det \overline{B}}{\det A}, \; \frac{d\widehat{\lambda}}{dr} = \frac{\det \widehat{B}}{\det A}, \; \frac{d\underline{\lambda}}{dr} = \frac{\det \underline{B}}{\det A},$$

where

$$A = \begin{bmatrix} U^H_{\overline{\lambda}} & U^H_{\underline{\lambda}} & U^H_{\widehat{\lambda}} \\ U^L_{\overline{\lambda}} & U^L_{\underline{\lambda}} & U^L_{\widehat{\lambda}} \\ U^V_{\overline{\lambda}} & U^V_{\underline{\lambda}} & U^V_{\widehat{\lambda}} \end{bmatrix}.$$

Matrices $\overline{B}$, $\underline{B}$ and $\widehat{B}$ are obtained by replacing, respectively, the first, second and third column of A with $[-\overline{U}_r, -\underline{U}_r, -\widehat{U}_r]'$. Observe that

$$\begin{aligned} U^H_{\overline{\lambda}} &= -U^H; \quad U^H_{\underline{\lambda}} = \overline{z}_0\widehat{z}_0\underline{z}_0(r - b); \quad U^H_{\widehat{\lambda}} = \overline{z}_0\widehat{z}_0\underline{z}_0(r - b) + \overline{z}_0\widehat{z}_0(b - v) \\ U^L_{\overline{\lambda}} &= U^L_{\widehat{\lambda}} = -U^L; \quad U^L_{\underline{\lambda}} = \overline{z}_0\widehat{z}_0\underline{z}_0(r - 1) - \overline{z}_0\widehat{z}_0\frac{1 - \underline{z}_0 - \underline{z}_1 - \underline{\lambda z}_1}{\underline{\lambda}}(1 - b) \\ U^V_{\overline{\lambda}} &= U^V_{\widehat{\lambda}} = -U^V; \quad U^V_{\underline{\lambda}} = \overline{z}_0\widehat{z}_0\underline{z}_0(r - b); \quad U^H_r = U^L_r = U^V_r = -\overline{z}_0\widehat{z}_0\underline{z}_0. \end{aligned}$$

It follows that

$$\det A = \left(U^H_{\widehat{\lambda}} + U^H\right)\left(U^V U^L_{\underline{\lambda}} - U^V_{\underline{\lambda}} U^L\right).$$

Observe that $U^H_{\widehat{\lambda}} + U^H$ is positive and that $U^L_{\underline{\lambda}} < U^V_{\underline{\lambda}} < 0$. In addition $U^L < U^V$ because $v > v^\star$ (Condition (24)). It follows that $\det A < 0$. Furthermore

$$\begin{aligned} \det \overline{B} &= \overline{z}_0\widehat{z}_0\underline{z}_0 \times \left(U^H_{\widehat{\lambda}} + U^H\right) \times \left(U^V_{\underline{\lambda}} - U^L_{\underline{\lambda}}\right) \\ \det \underline{B} &= \overline{z}_0\widehat{z}_0\underline{z}_0 \times \left(U^V - U^L\right) \times \left(U^H_{\widehat{\lambda}} + U^H\right) \\ \det \widehat{B} &= \overline{z}_0\widehat{z}_0\underline{z}_0 \times \left(U^H - U^V\right) \times \left(U^H_{\underline{\lambda}} - U^L_{\underline{\lambda}}\right). \end{aligned}$$

One can verify that $\det \overline{B}$, $\det \underline{B}$ and $\det \widehat{B}$ are all positive and independent of r. Substitute $U^H = \overline{U}$, $U^L = \underline{U}$ and $U^V = \widehat{U}$ into the $d\Pi/dr$ and rearrange to obtain

$$\frac{d\Pi}{dr} = \frac{\overline{z}_0 \widehat{z}_0}{\det A} \times (c_1 r - c_2),$$

where

$$c_1 = \underline{z}_0 \left(\det \overline{B} + \det \underline{B} + \det \widehat{B}\right) > 0$$
$$c_2 = (1-b)\left[(1-\underline{z}_0)\left(\det \overline{B} + \det \underline{B}\right) + \frac{1-\underline{z}_0-\underline{z}_1}{\underline{\lambda}} \det \underline{B}\right] > 0.$$

Solving the FOC for r we obtain

$$\frac{d\Pi}{dr} = 0 \Leftrightarrow r = \frac{c_2}{c_1} = \frac{\underline{\lambda}(1-b)(v-b)(1-\underline{z}_0-\underline{z}_1)}{(1-b)(\underline{z}_0-\underline{z}_0^2-\underline{z}_1)+(v-b)\underline{\lambda}\underline{z}_1} := r^\star$$

Notice that when buyers are identical in their valuations the reserve price approaches to $\widehat{r}$, that is $\lim_{v\to 1} r^\star = \widehat{r}$. To verify the SOC notice that $\overline{z}_0\widehat{z}_0/\det A < 0$; hence

$$sign\left(\frac{d\Pi}{dr}\right) = -sign\,(c_1 r - c_2).$$

Observe that c_1 and c_2 are both positive constants since $\det \overline{B}$, $\det \underline{B}$ and $\det \widehat{B}$ are all positive and independent of r. Therefore $d\Pi/dr > 0$ for all $r < r^\star$ and $d\Pi/dr < 0$ for all $r > r^\star$, which means that $r = r^\star$ is the global maximum. Finally note that Π is constructed under the conjecture that $r < b$; condition (24) requires $r^\star < b$ verifying the conjecture. ■

We complete this section by proving that this equilibrium cannot survive if sellers are allowed to trade via fixed pricing. The proof of the argument is fairly straightforward and follows the steps of the proof of Proposition 4.1.

Proposition 5.2 *The auction equilibrium is not sustainable if $\mathbb{M}$ includes price posting.*

Proof. Along the auction equilibrium we have $U^L\left(r^\star\right) < U^V\left(r^\star\right) < U^H\left(r^\star\right)$. Indeed $U^V < U^H$ is true for all r, whereas $U^L < U^V$ because $v > v^\star$. Furthermore notice that

$$r^\star > \frac{1-\underline{z}_0-\underline{z}_1}{\underline{z}_0}(1-b)$$

implying that $U^L < \overline{z}_0\widehat{z}_0\underline{z}_0$.

Now, consider a seller who switches to fixed pricing and posts some $r' < b$. Let U' denote the expected utility of buyers at the deviant store. Since the seller competes via fixed pricing U' is the same for all customers present at the store no matter what the type. Since $U^L < U^V < U^H$ it follows that the deviant store attracts low budget customers only. This is argument is made more precisely in the proof of Proposition 4.1. Hence, the queue lengths at the deviant store are as follows: $\overline{\lambda}' = \widehat{\lambda}' = 0$ and $\underline{\lambda}'$ satisfies the usual indifference condition. The seller's problem is

$$\max_{r'<b,\underline{\lambda}'\in\mathbb{R}_+} \left[1 - z_0\left(\underline{\lambda}'\right)\right] r' \;\; \text{subj. to} \;\; \frac{1-\underline{z}_0'}{\underline{\lambda}'}(1-r') = U^L\left(r^\star\right).$$

The FOC $z_0(\underline{\lambda}') = U^L$ implies that he posts $r' = r_f(\underline{\lambda}')$ and expects to earn

$$\Pi' = \pi(\underline{\lambda}') = 1 - \underline{z}_0' - \underline{z}_1'.$$

The FOC further implies that $\underline{\lambda}' = -\ln U\left(r^\bigstar\right)$; hence condition (24) guarantees that

$$r_f(\underline{\lambda}') = 1 + \frac{U^L\left(r^\bigstar\right)\ln U_L\left(r^\bigstar\right)}{1 - U_L(r^\bigstar)} < b,$$

which is what we have conjectured. Now we verify that Π' exceeds Π, which is given by 23. We have

$$\Pi' > \Pi \Leftrightarrow \overline{z}_0(1 - \widehat{z}_0)(1 - v) + \overline{z}_0\widehat{z}_0\underline{z}_0 + \underline{\lambda}U^L + \overline{\lambda}U^H + \widehat{\lambda}U^V - \underline{z}_0' - \underline{z}_1' > 0.$$

Since $U^L < U^V < U^H$ and $\overline{\lambda} + \underline{\lambda} + \widehat{\lambda} = \lambda$ it suffices to show

$$\Delta := \overline{z}_0(1 - \widehat{z}_0)(1 - v) + \overline{z}_0\widehat{z}_0\underline{z}_0 + \lambda U^L - \underline{z}_0' - \underline{z}_1' > 0.$$

Observe that $\overline{z}_0\widehat{z}_0\underline{z}_0 = e^{-\lambda}$. In addition the FOC $\underline{z}_0' = U^L$ implies that $\underline{\lambda}' > \lambda$ because $U^L < \overline{z}_0\widehat{z}_0\underline{z}_0$. Substitute $\underline{z}_0' = U^L$ into Δ to obtain

$$\Delta = \overline{z}_0(1 - \widehat{z}_0)(1 - v) + \underline{z}_0'(e^x - 1 - x),$$

where $x := \underline{\lambda}' - \lambda > 0$. The first expression is positive; the expression $e^x - 1 - x$ is also positive for all $x > 0$. Hence $\Delta > 0$ i.e. the deviation is profitable. ■

Observe that a deviation to fixed price trading is possible if buyers are sufficiently similar in terms of their valuations *and* budgets. Indeed condition (24) requires b as well as v to be large. This is is the same intuition that follows from Proposition 4.1.